

# $\eta/s$ at finite coupling

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## Abstract

We compute the ratio of the coefficient of shear viscosity to entropy density at finite coupling and at zero chemical potential using holographic duality, up to ten derivative terms in the low energy effective 5-dimensional action, of a specific kind, which may or may not be connected to the supersymmetric completion of Type IIB theory. The result suggests that this ratio can be positive only for the 8th derivative term even with the form of that term in the action as  $\mathcal{C}^{ij}_{kl}\mathcal{C}^{kl}_{mn}\mathcal{C}^{mn}_{rs}\mathcal{C}^{rs}_{ij}$ , where  $\mathcal{C}$  is the Weyl tensor.

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# 1 Introduction

It has become very interesting to understand the properties of the newly discovered state of matter, which is characterized by having very high energy density, strong collective flow and early thermalization time [1], at RHIC and may be called as the "first signature" for the formation of quark-gluon plasma. However, the strong criteria to see such a phase would be the quark deconfinement and chiral symmetry restoration [1]. Moreover, since the phase is seen at the strongly coupled limit means the quantitative study of this phase is going to be very cumbersome. However, we can understand the properties of some other theoretically constructed phases of similar kind, which may share some of its properties in common with sQGP. Let us assume that is the case and denote this theoretically constructed phase as hQGP. This means some universal properties could exist which are in common to both sQGP as well as to hQGP, i.e. those properties that remain the same irrespective of the model understudy. Now, we can understand quantitatively the properties of these phase using the holographic prescription [2].

One such quantity is the ratio of shear viscosity to entropy density,  $\eta/s$ . This ratio has been computed earlier for  $\mathcal{N} = 4$  system at finite temperature [3] and is found to have the value  $\frac{1}{4\pi}$ . It has been conjectured in [4] that for any (relativistic) system that admits a gravity dual should have a minimum value of  $\frac{1}{4\pi}$  at zero chemical potential. This conjecture was based mainly on the calculations done (during that time) in a very specific limit that is in the large rank of the gauge group and for large value of the 't Hooft coupling ( $N \rightarrow \infty$ ,  $\lambda \rightarrow \infty$ ) and also in the absence of enough experimental data. However, recently it has been shown explicitly that for some theoretically constructed models this ratio can even go below  $\frac{1}{4\pi}$  and this happens at finite 't Hooft coupling limit [5] and [6]. In [7], a theoretical argument was provided with the help of a Gedanken type experiments to falsify the KSS conjecture. See [8], for nice reviews on the subject.

In this note, we shall present results based on the calculations done using holographic duality. We evaluate the ratio of  $\eta/s$  up to tenth orders in derivative expansion using the elegant approach proposed by Iqbal and Liu [9] and is beautifully presented to study some more examples in [10]. In this approach [9], one important property is: the transport properties at the boundary is fully captured at the horizon but only in the zero frequency limit. This approach has the drawback, we can evaluate the transport quantities only in the scalar channel.

In our computation, the only degrees of freedom we have considered is the metric. It is assumed not to couple with any other degrees of freedom or with itself in a manner other than those that are governed by Weyl tensor, which means no derivative occurs to tensors like Ricci, Riemann tensors and Ricci scalar. This suggests the action will be constructed using the Weyl tensor and we choose the contraction of indices to be of a very specific kind.

The result of the calculation is that at the fourth power to Weyl tensor, of the type  $\mathcal{C}^{ij}_{kl}\mathcal{C}^{kl}_{mn}\mathcal{C}^{mn}_{rs}\mathcal{C}^{rs}_{ij}$ , we can see there is an enhancement to both the shear viscosity and the

entropy density and making the ratio of  $\eta/s$  to increase above  $1/4\pi$ , provided we take the corresponding coupling  $\lambda_4$  to be a positive number.

The paper is organized as follows: in section 2, we shall write down a low energy effective action in 5-dimension, whose  $\eta/s$  we want to calculate and review briefly the computation of  $\eta$  following [9] and entropy density using the Wald's prescription [11]. Then we will present the result of the computation in section 3. We have relegated the explicit computational details of both  $\eta$  and entropy density for two examples, the Gauss-Bonnet term and square of Weyl tensor in section 6 of Appendix A and the relation between  $\lambda_2$  with the central charges in section 7 of Appendix B.

The computation to  $\eta/s$  for different examples both with and without the higher derivative corrections have been studied in [10], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22] and [23].

## 2 The action and the prescription

Let us assume the low energy effective action that we are interested in has the following structure

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_2 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{ij} + \lambda_3 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{ij} + \lambda_4 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{rs} \mathcal{C}^{rs}_{ij} + \lambda_5 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{rs} \mathcal{C}^{rs}_{uv} \mathcal{C}^{uv}_{ij} \right], \quad (1)$$

where  $\mathcal{C}_{abcd}$  is the Weyl tensor. In order to evaluate the quantities of interest, we shall follow [9], which says that the shear viscosity can be evaluated by computing the following quantity

$$\eta = \lim_{k_a \rightarrow 0} \frac{\Pi(r, k_a)}{i\omega \phi(r, k_a)} \quad (2)$$

at the boundary, where  $\Pi$  is the momentum associated to the field  $\phi$  and is related to the metric fluctuation, in particular, to the fluctuation in scalar channel. Let us denote the metric fluctuation

$$h^y_x = \int [dk] \phi_k(r) e^{-i\omega t + ikz}, \quad (3)$$

where the graviton is moving along  $z$  direction and we are using a short hand notation to write the appropriate measure factor for momentum integrals and factors of  $2\pi$  in  $[dk]$ .

Considering this kind of fluctuation, one can show that the field  $\phi$  decouples from the rest of the metric fluctuation [24] and its equation of motion can be derived from the following effective action

$$S = \frac{1}{16\pi G} \int dr [dk] \left( A(r) \phi_k'' \phi_{-k} + B(r) \phi_k' \phi_{-k}' + C(r) \phi_k' \phi_{-k} + D(r) \phi_k \phi_{-k} + E(r) \phi_k'' \phi_{-k}'' + F(r) \phi_k'' \phi_{-k}' \right) + \mathcal{K}, \quad (4)$$

where  $\mathcal{K}$  is the appropriately generalized Gibbons-Hawking boundary term, which is written in [12],[10]. One of the important point is that upon inclusion of higher derivative terms to action, it generically gives us a non-trivial form of the functions  $E(r)$  and  $F(r)$ . For example, in the 2-derivative action, like the Einstein-Hilbert type, these two functions vanish and are non-trivial for other higher derivative terms like Gauss-Bonnet etc. This means these two functions  $E(r)$  and  $F(r)$  are already of order  $\lambda_i$ . As we know via the Kubo formula, the coefficient of shear viscosity is related to the two point correlation function involving the energy momentum tensor and our interest is to compute it to leading order in the parameters  $\lambda_i$ , means we shall compute the functions  $A, B, \dots, F, E$  etc., to linear in  $\lambda_i$  and not like the product of  $\lambda_i$ 's or powers of more than unity.

By considering the radial coordinate as time, as is considered in the paper of Liu et al., we can compute the momentum and it comes out as

$$\Pi_k(r) = \frac{1}{8\pi G} \left[ \left( B - A - \frac{F'}{2} \right) \phi'_k(r) - \frac{d}{dr} (E \phi''_k) \right]. \quad (5)$$

In order to compute the momentum in zero frequency limit and then evaluating the ratio eq(2) at the boundary requires us to know the derivatives of the field  $\phi$ . In the zero frequency limit the momentum is constant [9], which means it has the same value either on the boundary or at the horizon. On evaluating it at the horizon with the in falling boundary condition for the field  $\phi$  requires us to use

$$\partial_r \phi = -i\omega \sqrt{\frac{b}{a}} \phi, \quad \partial_r^n \phi = -i\omega \left( \partial_r^{n-1} \sqrt{\frac{b}{a}} \right) \phi, \quad (6)$$

which is argued to work only at the horizon in [9] and we have assumed the five-dimensional metric has the following form, where there is a rotational invariance in the spatial directions

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + c(r)[dx^2 + dy^2 + dz^2] \quad (7)$$

Using these ingredients that eq(2) gives us, after dropping the index  $k$ , of course the momentum dependencies are there in the field,  $\phi$

$$\eta = \lim_{\omega \rightarrow 0} \frac{\Pi}{i\omega \phi} = \frac{1}{8\pi G} [\kappa_2(r) + \kappa_i(r)]_{r=horizon}, \quad (8)$$

where

$$\kappa_2 = \sqrt{\frac{b}{a}} \left[ A - B + \frac{F'}{2} \right], \quad \kappa_i = \frac{d}{dr} \left[ E \frac{d}{dr} \sqrt{b/a} \right] \quad (9)$$

In order to evaluate the ratio  $\eta/s$ , we need to know the expression to entropy density  $s$ . The entropy of the gravitational system can be calculated using Wald's formula [11]

$$\mathcal{S} = -2\pi \int \left( \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \right)_{horizon}, \quad (10)$$

where  $L$  is the Lagrangian that follows from action eq(1) and the binormal quantity  $\epsilon_{ab}$  is normalized to obey  $\epsilon_{ab}\epsilon^{ab} = -2$ . One particular choice to construct such an object is to introduce two null vectors  $\xi_a$  and  $\delta_b$  with the restriction  $\xi.\delta = 1$  on the horizon [25]

$$\epsilon_{ab} = \xi_a \delta_b - \delta_a \xi_b, \quad (11)$$

with the choice to our metric as written in eq(7), it gives

$$\xi_t = -a, \quad \xi^t = 1, \quad \delta_t = 1, \quad \delta^t = -\frac{1}{a}, \quad \delta_r = -\sqrt{\frac{b}{a}}, \quad \delta^r = -\frac{1}{\sqrt{ab}}, \quad (12)$$

and the rest of the components of  $\xi_a$  and  $\delta_b$  vanishes.

### 3 Computation

Before doing the computation of  $\eta$  for non-zero but small positive  $\lambda_i$ , let us write down the solution to metric for the case  $\lambda_i = 0$ . This solution can be read out very easily from the paper [5] and is given by

$$a = N^2 \frac{(r^4 - r_0^4)}{L^2 r^2}, \quad b = \frac{L^2 r^2}{r^4 - r_0^4}, \quad c = \frac{r^2}{L^2}. \quad (13)$$

Let us set  $N = 1$  or can be re-absorbed in defining the temporal coordinate. Now, to do the computation for both  $\eta$  and  $\mathcal{S}$ , we do not need to know the  $\lambda_i$  corrected solution<sup>1</sup>. This is due to the reason that we are only interested to calculate these quantities at the leading order in  $\lambda_i$  only and not in their product. This also means that we can calculate each of the higher derivative terms without the need to worry about the other terms in the action. It means the action that we need to consider are the Einstein-Hilbert term, the cosmological constant term and any one of the higher derivative term to find the result at that order in derivative expansion. For example, the action that we shall start out with contains only the  $\lambda_2$  term

$$S_2 = \frac{1}{16\pi G} \int \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_2 C^{ij}_{kl} C^{kl}_{ij} \right] \quad (14)$$

Going through the above mentioned procedure as outlined in section 2, this gives us the result that there is an enhancement to both the coefficient of shear viscosity and to the entropy density ( assuming  $\lambda_i$  is positive, which we do not claim it should always be. )

$$\eta = \left( \frac{r_0^3}{16\pi G L^3} \right) \left[ 1 + \frac{4\lambda_2}{L^2} \right], \quad s \equiv \frac{\mathcal{S}}{V_3} = \left( \frac{r_0^3}{4GL^3} \right) \left[ 1 + 12 \frac{\lambda_2}{L^2} \right] \quad (15)$$

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<sup>1</sup>This we demonstrate by going through two different examples in the appendix. The main point is that the the uncorrected piece to both  $\eta$  and entropy density depends only on the spatial part of the metric component, which do not receive any correction at the linear order in  $\lambda_i$ .

giving the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - 8 \frac{\lambda_2}{L^2} \right]. \quad (16)$$

We do not write down explicitly the form of  $\kappa_2$  and  $\kappa_4$  as these are very big expressions and similarly for other cases too.

At this order, if we would have considered the Gauss-Bonnet term instead of the square of the Weyl tensor term then the result on both the coefficient of shear viscosity and the entropy density would have been different. In this case there occurs a suppression to  $\eta$  and unchanged form of entropy density [5]

$$\eta = \left( \frac{r_0^3}{16\pi GL^3} \right) \left[ 1 - \frac{8\lambda_2}{L^2} \right], \quad s \equiv \frac{\mathcal{S}}{V_3} = \left( \frac{r_0^3}{4GL^3} \right), \quad (17)$$

but the ratio remains the same

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - 8 \frac{\lambda_2}{L^2} \right], \quad (18)$$

as that considered by including the square of Weyl term.

Let us move to the next order in derivative expansion and consider the following action

$$S_3 = \frac{1}{16\pi G} \int \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_3 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{ij} \right] \quad (19)$$

The computation results in the following form to the shear viscosity and the entropy density

$$\eta = \left( \frac{r_0^3}{16\pi GL^3} \right) \left[ 1 - \frac{336\lambda_3}{L^4} \right], \quad s \equiv \frac{\mathcal{S}}{V_3} = \left( \frac{r_0^3}{4GL^3} \right) \left[ 1 + 48 \frac{\lambda_3}{L^4} \right], \quad (20)$$

and the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - 384 \frac{\lambda_3}{L^4} \right]. \quad (21)$$

If we would have taken another structure to the action at this order instead of considering the cubic power to Weyl tensor, like the one that follows from the third order term to Lovelock gravity

$$\begin{aligned} S_3 = & \frac{1}{16\pi G} \int \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_3 \left( 2R^{ijkl} R_{klmn} R^{mn}_{ij} + 8R^{ij}_{kl} R^{km}_{jn} R^{ln}_{im} + \right. \right. \\ & 24R^{ijkl} R_{kljn} R^n_i + 3RR^{ijkl} R_{klj} + 24R^{ijkl} R_{ki} R_{lj} + 16R^{ij} R_{jk} R^k_i - \\ & \left. \left. 12RR^{ij} R_{ij} + R^3 \right) \right], \end{aligned} \quad (22)$$

then the shear viscosity is same as that of computing from a 2-derivative action i.e the Einstein-Hilbert action. This happens because the momentum associated to the metric fluctuation vanishes, when evaluated at the horizon.

Let us proceed further and do the calculation by including yet another term to action, which is higher than the previous one that is the fourth power of Weyl tensor. The explicit form the action is

$$S_4 = \frac{1}{16\pi G} \int \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_4 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{rs} \mathcal{C}^{rs}_{ij} \right]. \quad (23)$$

The computation to shear viscosity and entropy density at this order results

$$\eta = \left( \frac{r_0^3}{16\pi G L^3} \right) \left[ 1 + \frac{1440\lambda_4}{L^6} \right], \quad s \equiv \frac{\mathcal{S}}{V_3} = \left( \frac{r_0^3}{4GL^3} \right) \left[ 1 + 480 \frac{\lambda_4}{L^6} \right], \quad (24)$$

with the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + 960 \frac{\lambda_4}{L^6} \right]. \quad (25)$$

So, we see that for the fourth power to Weyl tensor both the coefficient of shear viscosity and the entropy density increases only for positive  $\lambda_4$ . Thus making the ratio to increase and respects the KSS bound.

In literature the computation at this order is done by including the other kind of contraction of indices [12] and the result is that ratio  $\eta/s$  respects the KSS bound. Here we see the result of respecting the KSS bound follows just by considering only one kind of contraction to indices.

Let us proceed further and include the next order term that is a fifth power of Weyl tensor, with the structure to action

$$S_5 = \frac{1}{16\pi G} \int \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_5 \mathcal{C}^{ij}_{kl} \mathcal{C}^{kl}_{mn} \mathcal{C}^{mn}_{rs} \mathcal{C}^{rs}_{uv} \mathcal{C}^{uv}_{ij} \right] \quad (26)$$

The computation results at this order to shear viscosity and entropy density as

$$\eta = \left( \frac{r_0^3}{16\pi G L^3} \right) \left[ 1 - \frac{7040\lambda_5}{9L^8} \right], \quad s \equiv \frac{\mathcal{S}}{V_3} = \left( \frac{r_0^3}{4GL^3} \right) \left[ 1 + 3200 \frac{\lambda_5}{L^8} \right], \quad (27)$$

with the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - 35840 \frac{\lambda_5}{9L^8} \right]. \quad (28)$$

From the result it just follows that even though there is an enhancement to entropy density but the suppression to shear viscosity makes the ratio  $\eta/s$  to go below the KSS bound, with the assumption that  $\lambda_5$  is positive.

Now we can include all the independent contributions and write the ratio to  $\eta/s$  as

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 - 8 \frac{\lambda_2}{L^2} - 384 \frac{\lambda_3}{L^4} + 986 \frac{\lambda_4}{L^6} - \frac{35840}{9} \frac{\lambda_5}{L^8} \right], \quad (29)$$

which we can always do because we are interested to compute the ratio only to leading order in  $\lambda_i$  and not their product.

We can compare our result with the one that appeared recently [13] and the result up to the linear order in  $\lambda_3$  matches and the coefficient of  $\lambda_4$  and  $\lambda_5$  are new. In the paper [12], the authors had done the calculation by considering a  $\mathcal{C}^4$  term in the action but the way the indices are contracted and the number of such terms are different than we have considered in the present paper. The reason of considering such a single term at the  $\mathcal{C}^4$  order is just to show that even with one term we can get a positive contribution to the ratio  $\eta/s$ .

From the result of the computation eq(29), it just follows that to leading in  $\lambda_i$ , we can rewrite it as

$$\eta/s = \frac{1}{4\pi} \left[ 1 - \sum_{i=2} \mathfrak{d}_i \frac{\lambda_i}{L^{2(i-1)}} \right], \quad (30)$$

where  $\mathfrak{d}_i$  is a constant, which can take both positive and negative values but the magnitude of it increases with the number of derivatives that we are considering in the low energy effective action. From dimensional analysis, it just follows that  $\lambda_i \sim \alpha'^{(i-1)}$ , where  $\alpha' = l_s^2$ , the square of string length, which by AdS/CFT duality means  $\lambda_i \sim \frac{1}{\lambda^{\frac{i-1}{2}}}$ , where  $\lambda$  is the 't Hooft coupling.

As a check to  $\mathfrak{d}_2$  and  $\mathfrak{d}_3$ , we find that our computation matches with the ones reported in [13].

## 4 Conclusion and discussion

Using the approach of [9], we have calculated one of the transport quantity that is  $\eta/s$  and have found some agreement with the results mentioned in [13] and predicted a couple more. In particular, we have shown that the KSS bound is respected by considering one particular type of term at the fourth power to Weyl tensor, provided we take the coupling  $\lambda_4$  as a positive quantity. To the fifth power of Weyl tensor, the ratio  $\eta/s$  do not respect the KSS bound, once again for positive coupling  $\lambda_5$ . It is expected on general grounds that the couplings  $\lambda_i$  in the gravity side are related to the central charges  $a$  and  $c$  on the dual field theory but the precise relation is not known for all  $\lambda_i$ , at present. However,  $\lambda_2$  is related to  $a$  and  $c$  via eq(50)

$$\frac{\lambda_2}{L^2} = \frac{1}{8} \frac{c - a}{a}, \quad (31)$$

which are further related to two parameters  $t_2$  and  $t_4$  defined in [26] with some constraints on the parameters  $t_2$  and  $t_4$  coming from the argument that there should be only positive energy that is deposited on the calorimeter "experiment". There exists various restrictions on  $a/c$ , depending on the amount of supersymmetry preserved by the system [26]. For a system that does not preserve any amount of supersymmetry, one gets the restriction on  $\lambda_2$  as  $-\frac{13}{248} \leq \frac{\lambda_2}{L^2} \leq \frac{1}{4}$ , for  $\mathcal{N} = 1$ , it is  $-\frac{1}{24} \leq \frac{\lambda_2}{L^2} \leq \frac{1}{8}$  and for  $\mathcal{N} = 2$ , it is  $-\frac{1}{40} \leq \frac{\lambda_2}{L^2} \leq \frac{1}{4}$ .



Certainly, it is interesting to use the criteria of micro-causality violation [14] to find, if there exists any other constraint on the  $\lambda_i$ 's. But we do not have the explicit gravity solutions for all these cases, which deserve further investigations and possible violation to KSS bound[6].

One of the important question that arises in the study is the convergence of eq(30) i.e how small are the  $\lambda_i$ 's such that the sum in eq(30) converges ? Is the convergence going to be a criteria to fix the nature of the action other than supersymmetry?

The choice of taking, the kind of action as written in eq(1) may not have been inspired from string theoretic suggestions, especially the 8th derivative term. If we take the string theoretic suggestions at this order then we already know the answer to  $\eta/s$  [12] and the aim here is not to repeat the calculation. Rather to show that with a specific kind of contraction to indices at  $\mathcal{C}^4$  order in derivative expansion, we can still make the KSS bound to hold provided the coupling  $\lambda_4$  is positive. Of course, a priori there is not any reason to accept this kind of low energy effective action. It could be, one may turn around and ask the question: Is it possible to construct any other type of action at this order in the derivative expansion, such that it respect KSS bound ? And here is an example. Also, we are doing phenomenology and trying to find the consequences, if the low energy effective action has a form the kind eq(1).

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## 6 Appendix A

In this appendix, we shall demonstrate why we do not need to know the exact form of the  $\lambda_i$  corrected solution, by going through two different examples, as stated in the main text. First we shall show for the case, when the gravitational action has two parts: one is Einstein-Hilbert action and the other is of the Gauss-Bonnet kind. Then we shall consider adding the Weyl-squared term. The result is that as long as the spatial part of the metric components do not receive any corrections to linear in  $\lambda_i$ , the computation of  $\eta/s$  do not require the  $\lambda_i$  corrected solution at this order.

## 6.1 Einstein-Hilbert and Gauss-Bonnet term

In this case the action is described by

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_2 (R_{ijkl} R^{ijkl} - 4R_{ij} R^{ij} + R^2) \right] \quad (32)$$

and it admits an exact solution of the following form

$$ds^2 = -r^2 \alpha^2(r) dt^2 + \frac{dr^2}{r^2 \beta^2(r)} + \gamma^2(r) (dx^2 + dy^2 + dz^2). \quad (33)$$

We have checked explicitly that this form of the metric solves the equations of motion and the form of the metric components are [5]

$$\alpha^2(r) = \beta^2(r) = \frac{1}{4\lambda_2} \left[ 1 - \sqrt{1 - 8\lambda_2 \left( 1 - \frac{r_0^4}{r^4} \right)} \right], \quad \gamma^2(r) = r^2, \quad (34)$$

where we have set the size of AdS radius to unity, for convenience. As reviewed in section 2, we can calculate the coefficient of shear-viscosity using eq(8). After completing the necessary computations to find the expressions for  $A$ ,  $B$ ,  $E$  and  $F$ , we ended up

$$\begin{aligned} \frac{A}{16\pi G} &= \frac{\sqrt{abc^3}}{8\pi G ab^2 c^2} [abc^2 - \lambda_2 (2a'cc' + ac'^2)], \\ \frac{B}{16\pi G} &= \frac{\sqrt{abc^3}}{8\pi G a^2 b^3 c^2} [3a^2 b^2 c^2 + \lambda_2 (bc^2 a'^2 + 3ac^2 a'b' - 11abca'c' + 3a^2 cb'c' - 5a^2 bc'^2 \\ &\quad - 2abc^2 a'' - 2a^2 bcc'')], \\ \frac{F}{16\pi G} &= -\lambda_2 \frac{\sqrt{abc^3}}{8\pi G ab^2 c} [ca' + ac'], \\ \frac{E}{16\pi G} &= 0, \end{aligned} \quad (35)$$

where  $a$ ,  $b$  and  $c$  are the metric components that appear in the notation of eq(7). The explicit form of

$$\frac{\kappa_2}{16\pi G} = \frac{c^{3/2}}{32\pi G} - \lambda_2 \frac{\sqrt{ca'c'}}{32\pi G ab}, \quad \kappa_i = 0, \quad (36)$$

with

$$a = b^{-1} = \frac{r^2}{4\lambda_2} \left[ 1 - \sqrt{1 - 8\lambda_2 \left( 1 - \frac{r_0^4}{r^4} \right)} \right], \quad c = r^2. \quad (37)$$

Computing  $\eta$  using eq(8) and keeping terms to linear in  $\lambda_2$ , gives the result as written in eq(17). The main point here is that the term  $\kappa_2$  and  $\kappa_i$  upon evaluating at the horizon,

have the same structure whether we use the  $\lambda_2$  corrected solution or not. This essentially says that the extra terms that contribute to  $\kappa_2(r)$  vanish upon evaluating at the horizon.

The entropy density calculated using the Wald's formula give the same answer whether we use the  $\lambda_2$  corrected solution eq(33) or the uncorrected solution eq(13). This is very easy to convince oneself. The reason behind this is that there appears two terms after the differentiation in eq(10), one from differentiating the Einstein-Hilbert term and the other is from differentiating the Gauss-Bonnet term. The second kind of term is already linear in  $\lambda_2$ , and our interest is to calculate entropy to linear order, which means the uncorrected solution eq(13) is good enough for this term. Whereas the first term depends on the spatial part of the metric components  $\gamma^2(r)$ , which do not receive any corrections in  $\lambda_2$ , the computation of entropy density too does not require the  $\lambda_2$  corrected solution.

## 6.2 Einstein-Hilbert and Weyl-squared term

In this case the action is described by

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \lambda_2 \mathcal{C}_{ijkl} \mathcal{C}^{ijkl} \right] \quad (38)$$

The equations of motion that results

$$\begin{aligned} R_{ij} - \frac{1}{2}g_{ij}R - \frac{\lambda_2}{2}g_{ij}\mathcal{W} - 6g_{ij} + \lambda_2 \left[ \frac{R}{3}R_{ij} + 2R_{ikpq}R_j{}^{kpq} + \frac{4}{3}R_{ipjs}R^{ps} - \right. \\ \left. 4R_{ip}R_j{}^p - \frac{1}{2}\nabla_i\nabla_j R - \frac{1}{2}\nabla_j\nabla_i R - \frac{1}{3}g_{ij}\nabla^2 R + \frac{4}{3}\nabla^2 R_{ij} \right] = 0, \end{aligned} \quad (39)$$

where  $\mathcal{W} = \mathcal{C}_{ijkl}\mathcal{C}^{ijkl}$  is the Weyl-squared term.

This equation of motion admits the following solution

$$ds^2 = -r^2\alpha^2(r)dt^2 + \frac{dr^2}{r^2\beta^2(r)} + \gamma^2(r)(dx^2 + dy^2 + dz^2). \quad (40)$$

Generically, it is very difficult to find the exact solution, however for our purpose the solution to linear in  $\lambda_2$  is good enough and is

$$\alpha = \beta = \frac{\sqrt{r^4 - r_0^4}}{r^2} \left[ 1 - \lambda_2 \left( \frac{r_0}{r} \right)^4 \right] + \mathcal{O}(\lambda_2)^2, \quad \gamma^2(r) = r^2 + \mathcal{O}(\lambda_2)^2. \quad (41)$$

It is interesting to note that  $\gamma$  do not receive any correction to linear in  $\lambda_2$ . Once, again going through the procedure as presented in section 2, gives

$$A = \frac{\sqrt{abc^3}}{48a^4b^4c^4} [96a^4b^3c^4 + \lambda_2(-16a^2b^2c^4a'^2 - 16a^3bc^4a'b' -$$

$$\begin{aligned}
& 16a^3b^2c^3a'c' + 16a^4bc^3b'c' + 32a^4b^2c^2c'^2 + 32a^3b^2c^4a'' - 32a^4b^2c^3c''), \\
B &= \frac{\sqrt{abc^3}}{48a^4b^4c^4} [72a^4b^3c^4 + \lambda_2(12a^2b^2c^4a'^2 + 12a^3bc^4a'b' + 16a^4c^4b'^2 - \\
& 52a^3b^2c^3a'c' - 44a^4bc^3b'c' + 56a^4b^2c^2c'^2 + 8a^3b^2c^4a'' - 8a^4b^2c^3c''), \\
F &= \lambda_2 \frac{\sqrt{abc^3}}{48a^4b^4c^4} [-32a^3b^2c^4a' - 64a^4bc^4b' + 96a^4b^2c^3c'], \\
E &= 4\lambda_2 \frac{\sqrt{abc^3}}{3b^2}.
\end{aligned} \tag{42}$$

Substituting all these terms into eq(8), gives

$$\eta = \frac{c^{3/2}}{16\pi G} + \lambda_2 \frac{\sqrt{c}}{32\pi G a^2 b^2} [bca'^2 + aca'b' - aba'c' - a^2b'c' - 2abca'' + 2a^2bc''], \tag{43}$$

which need to be evaluated at the horizon,  $r = r_0$ . The second piece of this equation is already linear in  $\lambda_2$ . From the first term, there will not be any correction at the linear order as  $c$  does not receive any correction at this order. The argument given in the previous subsection goes through for the calculation of entropy density and it gives the result as written in eq(15).

## 7 Appendix B: $a$ and $c$ central charges

Following [27], we now write down the expressions to central charges  $a$  and  $c$ . For the gravity action

$$S = \frac{1}{2\kappa_5^2} \int d^5x [R - 2\Lambda + \alpha R^2 + \beta R^{MN} R_{MN} + \gamma R_{MNKL} R^{MNKL}], \tag{44}$$

the central charges are

$$\begin{aligned}
\frac{c}{16\pi^2} &= \frac{L^3}{\kappa_5^2} \left[ \frac{1}{16} + \frac{1}{L^2} \left( -\frac{5\alpha}{2} - \frac{\beta}{2} + \frac{\gamma}{4} \right) \right], \\
\frac{a}{16\pi^2} &= \frac{L^3}{\kappa_5^2} \left[ \frac{1}{16} + \frac{1}{L^2} \left( -\frac{5\alpha}{2} - \frac{\beta}{2} - \frac{\gamma}{4} \right) \right],
\end{aligned} \tag{45}$$

where  $L$  is the size of the AdS radius. For a choice like that of the Gauss-Bonnet combination, namely,  $\alpha = t_2$ ,  $\beta = -4t_2$ ,  $\gamma = t_2$ , gives

$$\frac{c}{16\pi^2} = \frac{L^3}{16\kappa_5^2} \left[ 1 - 4\frac{t_2}{L^2} \right], \quad \frac{a}{16\pi^2} = \frac{L^3}{16\kappa_5^2} \left[ 1 - 12\frac{t_2}{L^2} \right]. \tag{46}$$

The ratio

$$\frac{a}{c} = \frac{1 - 12\frac{t_2}{L^2}}{1 - 4\frac{t_2}{L^2}} = 3 - \frac{2}{1 - 4\frac{t_2}{L^2}}. \tag{47}$$

This is the exact result, however, if we want to make contact with eq(5.1) of [5], then we need to identify  $\frac{t_2}{L^2} = \lambda_{GB}/2$  and to leading order in  $\lambda_{GB}$ ,

$$\begin{aligned}\frac{c}{16\pi^2} &= \frac{L^3}{16\kappa_5^2} \left[ 1 - 4 \frac{\lambda_{GB}}{2} \right] \simeq \frac{L^3}{16\kappa_5^2} \left[ \sqrt{1 - 4\lambda_{GB}} \right], \\ \frac{a}{16\pi^2} &= \frac{L^3}{16\kappa_5^2} \left[ 1 - 12 \frac{\lambda_{GB}}{2} \right] \simeq \frac{L^3}{16\kappa_5^2} \left[ 3 \sqrt{1 - 4\lambda_{GB}} - 2 \right], \\ \frac{a}{c} &\simeq 3 - \frac{2}{\sqrt{1 - 4\lambda_{GB}}}.\end{aligned}\tag{48}$$

For a choice like that of Weyl-squared combination,  $\alpha = \frac{t_2}{6}$ ,  $\beta = -\frac{4}{3}t_2$ ,  $\gamma = t_2$ , gives

$$\frac{c}{16\pi^2} = \frac{L^3}{16\kappa_5^2} \left[ 1 + 8 \frac{t_2}{L^2} \right], \quad \frac{a}{16\pi^2} = \frac{L^3}{16\kappa_5^2}.\tag{49}$$

It just follows trivially that

$$\frac{t_2}{L^2} = \frac{1}{8} \frac{c - a}{a}.\tag{50}$$

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